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Numerical methods-VIII

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[13-A-04] A scale-invariant third-order WENO scheme with optimal accuracy

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A scale-invariant third-order WENO scheme with optimal accuracy

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1 Introduction

Despite improvement over decades, there are still at least two key issues not fully solved in Weighted Essentially Non-oscillatory (WENO) schemes. Firstly, most WENO schemes hardly satisfy the scale-invariant property, which undermines their robustness and accuracy when simulating flows involving different scales such as reactive multi-component flows and plasma. Secondly, most of existing Essentially Non-oscillatory (ENO) constraints are not sufficient and necessary conditions to achieve optimal accuracy. Therefore, in this work, we design a new family of WENO schemes based on a three-cell stencil to address these two issues. The new WENO non-linear weights are designed as bounded asymptotic functions which are dependent on cell Normalized Volume Integrated Average (NVIA) values and a shape parameter θ. Thus, the new WENO weighting functions are scale-independent. The shape parameter θ controls the errors between the nonlinear weight and the ideal weight. As the value of θ increases, the new weighting functions become more accurate but also become more likely to produce numerical oscillations. Thus, we propose a downwind limiting method as a supplementary constraint of ENO property. The maximum value of θ which corresponds to the optimal accuracy of a weighting function can then be determined. The accuracy analysis via the standard Taylor expansion at critical points reveals that the proposed weighting functions outperform classical WENO schemes. The accuracy, scale-invariant property and ENO property of proposed schemes are further validated through benchmark tests, which shows the superior performance of the new schemes in comparison with existing three-cell based WENO schemes.

2 A New Type of Third Order WENO Scheme

2.1 Design of Scale-invariant WENO Schemes

Before proceeding further, we introduce some important definitions.

Definition 1. *With the initial VIA in a compact stencil* $S_i = \{I_{i-1}, I_i, I_{i+1}\}$*, the left-side-biased Normalised Volume Integrated Average (NVIA) value* $\hat{\phi}_i^L$ *over cell* $\mathcal{I}_i$ *is defined as*

$$\hat{\phi}_i^L = \frac{\bar{\phi}_i - \bar{\phi}_{i-1}}{\bar{\phi}_{i+1} - \bar{\phi}_{i-1}} \tag{1}$$

Definition 2. *A WENO operator is scale-invariant if the WENO operator* $\mathcal{W}[\cdot]$ *satisfies* $\mathcal{W}[\lambda\bar{\phi}] = \lambda\mathcal{W}[\bar{\phi}]$ *for any given* $\bar{\phi}$ *and scaling factor* $\lambda \in \mathbb{R}$ *(*$\lambda \neq 0$*).*

As defined and proved in [1], a WENO operator satisfies scale-invariant property, if and only if the weighting functions ω_j are independent of the scaling factor λ. Strictly, the existing WENO schemes only satisfy the scale-invariant property for a sufficiently large λ.

Proposition 1. *The WENO operator* $\mathcal{W}^L[\cdot]$ *is scale-invariant if the non-linear weights are only a function of NVIA.*

Thus, instead of relying on conventional smoothness indicators, we design a new family of weighting functions with normalised variable NVIA to achieve the scale-invariant property. The weighting functions we proposed are

$$w_0 = \frac{1}{3} + \frac{2}{3}\frac{1}{(1+\theta(\hat{\phi}_i^L)^{2m})^n} - \frac{1}{3}\frac{1}{(1+\theta(1-\hat{\phi}_i^L)^{2m})^n}, \quad m, n \in N^*, \theta > 0. \tag{2}$$

$$w_1 = 1 - w_0, \tag{3}$$

where θ is a shape parameter. Since the weighting functions are in the form of GMQ (General Multi-Quadric) functions, we denote the new WENO schemes as WENO-GMQ. With given positive integer m and n, we denote the proposed weighting function as $\omega_j^{(m,n)}$.

2.2 New Essentially Non-oscillatory Constraints with Optimal Accuracy

Here, we propose a downwind-limiting method as an additional constraint to achieve the essentially non-oscillatory property. The first order downwind scheme $\phi^L_{i+\frac{1}{2}} = \bar{\phi}_{i+1}$ can be retrieved by

$$\omega_0 = \frac{1 - \hat{\phi}_i^{\,L}}{2\hat{\phi}_i^{\,L} - 1}. \tag{4}$$

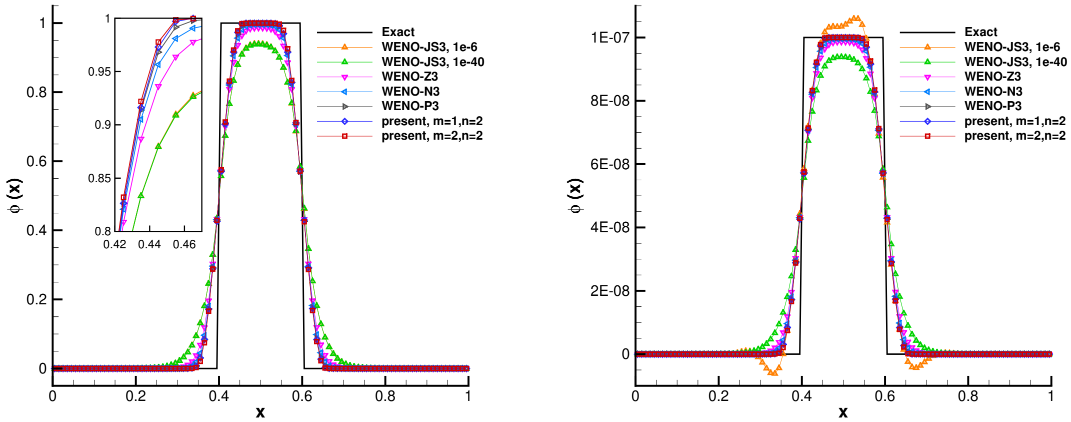
We denote the weighting function retrieving the first order downwind scheme by ω_j^D. Since the downwind scheme is not stable, we limit the information from the downwind cell in the reconstruction process. Thus, the maximum value of θ is determined by letting the following equation have exactly two real roots

$$\omega_0^{(m,n)} - \omega_0^D = 0. \tag{5}$$

3 Numerical results

3.1 Transportation of a square wave

We investigate the non-oscillatory property of the proposed scheme by transporting a single square wave. As shown in Fig. 1 (a), the present scheme obtains the least dissipative solution near the discontinuity. The new WENO scheme with $\omega_0^{(2,2)}$ presents slightly better numerical results than the one with $\omega_0^{(1,2)}$. It is also observed that, though WENO-JS3 with $\epsilon = 10^{-6}$ obtains almost the same result as that with $\epsilon = 10^{-40}$ with scaling factor $\lambda = 1$, WENO-JS3 with recommended $\epsilon = 10^{-6}$ generates numerical oscillations when the initial profile is scaled by $\lambda = 10^{-7}$ as shown in Fig. 1 (b).



(a) scaling factor $\lambda = 1$

(b) scaling factor $\lambda = 10^{-7}$

Figure 1: Numerical solution for advection of a square profile at time $t = 2$ on 100 uniform cells. (a) Initial condition with the range $[0, 1]$; (b) Initial condition is scaled by $\lambda = 10^{-7}$.

References

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